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Bifurcation solutions of cylindrically symmetric Yang–Mills equations

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Abstract. It is shown that even in the simplest case, when Yang–Mills equations with boundary conditions admit the Coulomb solution, there is a possibility for non-trivial, non-Abelian solutions. Bifurcation theory methods are used for finding these solutions.

1. Introduction

In this paper we will study the cylindrically symmetric solutions to the classical Yang–Mills field equations in the presence of an external static source. The Dirichlet boundary conditions will be imposed, but the Neumann boundary problem may be treated analogously.

In general one cannot expect that boundary conditions will give a unique solution to a nonlinear equation (although it is possible for particular classes of equations), even locally (in a suitable Banach space). This work will illustrate this. The use of bifurcation theory methods gives solutions with unexpected features. Even the Abelian pure electric field may have bifurcating non-Abelian fields with non-zero magnetic part.

The paper is organised as follows. In § 2 we discuss the cylindrically symmetric Yang–Mills equations (Sikivie and Weiss 1978) and the bifurcation theory methods are briefly reviewed. In § 3 will be presented an application of an analytical bifurcation method in the case when the Coulomb solution is admissible. General expressions for bifurcating potentials will be given. Section 4 contains one simple example which shows the physical nature of the obtained solutions. Section 5 includes some proposals about other uses of the bifurcation theory in Yang–Mills theory.

2. Review of the theory. Computations

Following Sikivie and Weiss (1978) we assume that the SU(2) Yang–Mills potentials $\boldsymbol{A}_\mu(\boldsymbol{x})$ are time independent, and

$$\boldsymbol{A}_0 = \boldsymbol{\phi}(\rho, x_3) \tag{1a}$$

$$\boldsymbol{A}_i = \epsilon_{i3j} x_j (\boldsymbol{A}(\rho, x_3)/\rho) \tag{1b}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$. Our interest is in $\boldsymbol{A}_0, \boldsymbol{A}_i$ of (Hölder) class $C^{2+\mu}$ at least $(0 < \mu \leq 1)$.

After a little computation, and putting $\phi^1=\phi^2=A^2=0$, one obtains (Sikivie and Weiss 1978)

$$-\nabla^2\phi(\rho,x_3)+g^2A^2(\rho,x_3)\phi(\rho,x_3)=b(\boldsymbol{x}) \tag{2a}$$

$$(\nabla^2-1/\rho^2)A(\rho,x_3)+g^2A(\rho,x_3)\phi^2(\rho,x_3)=0 \tag{2b}$$

where $A\equiv A^1$, $\phi=\phi^3$ and all other A^a, ϕ^a vanish, and $b(\boldsymbol{x})$ is the source of class $C^{0+\mu}$ at least.

Suppose that we know one solution A_0, ϕ_0 to equations ($2a, b$) satisfying certain boundary (Dirichlet, Neumann) conditions. Assume then

$$\phi\equiv\phi_0+B \qquad A\equiv A_0+\hat{A} \tag{3}$$

and insert these into equations (2).

Hence we obtain a sourceless system of equations with unknown functions B, $\hat{A}$ and with homogeneous boundary conditions:

$$-\nabla^2B+2g^2A_0\phi_0\hat{A}+g^2A_0^2B+g^2\hat{A}^2\phi_0+2g^2A_0\hat{A}B+g^2\hat{A}^2B=0$$
$$(\nabla^2-1/\rho^2+g^2\phi_0^2)\hat{A}+2g^2\phi_0A_0B+2g^2\phi_0\hat{A}B+g^2A_0B^2+g^2B^2\hat{A}=0. \tag{4}$$

The bifurcation phenomena are closely related to the so-called zero-mode solution (see e.g. Jackiw (1977)) fulfilling homogeneous boundary conditions. The existence of such a zero-mode solution is a necessary, but not sufficient, condition for the bifurcation.

In other words: if the equations (4) linearised at $B=0, \hat{A}=0$ admit non-trivial solutions at certain values of a parameter (i.e. g^2) then they may have a non-trivial solution $\hat{A}, B\neq 0$ in a neighbourhood of (0, 0) (Berger 1977, Krasnosel'skii 1964, Nirenberg 1974, Vainberg and Trenogin 1974).

We shall explain briefly why we need more information for the sufficiency. Let us denote by F the operator of equations (4) which is linearised at $\hat{A}=B=0$ and by T the nonlinear (in B, $\hat{A}$) operator of equations (4). Equations (4) can be written as

$$F(g^2)\begin{pmatrix}B\\ \hat{A}\end{pmatrix}+T(\hat{A},B,g^2)\begin{pmatrix}B\\ \hat{A}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{5}$$

Suppose f belongs to coker $F(g_0^2)=\ker F^*(g_0^2)$, where $F^*(g_0^2)$ is L_2-adjoint to $F(g_0^2)$. Multiply (5) by f^{T} (f^{T} is the transpose of f) and integrate over the domain. Thence

$$\int_\Omega f^{\mathrm{T}}(F(g^2)-F(g_0^2))\begin{pmatrix}B\\ \hat{A}\end{pmatrix}\mathrm{d}V+\int_\Omega f^{\mathrm{T}}T(\hat{A},B,g^2)\begin{pmatrix}B\\ \hat{A}\end{pmatrix}\mathrm{d}V=0. \tag{6}$$

So equation (6) gives a restriction upon the space of solutions of equations (5) (that is the origin of the Lyapunov–Schmidt equations—see below). We see now the importance of ideas such as index, ker, coker of operators. Note that, because of the ellipticity of F in our case, index, ker, coker are well defined (ker and coker are surely finite and, moreover, equal).

Topological methods give sufficient criteria for the existence of bifurcation (Berger 1977, Krasnosel'skii 1964, Nirenberg 1974), but our interest is not only in knowing that a bifurcating solution exists, but also in constructing it. Because of that we use an analytical method and the Lyapunov–Schmidt theorem (Vainberg and Trenogin 1974).

The operator F defined above is

$$F(g^2)=\begin{pmatrix} -\nabla^2+g^2A_0^2, & 2g^2A_0\phi_0 \\ 2g^2\phi_0A_0, & \nabla^2-1/\rho^2+g^2\phi_0^2 \end{pmatrix}. \tag{7}$$

F is Hermitian, so $\ker F=\operatorname{coker} F$, and the analytical index is equal to zero, $\operatorname{ind} F=\ker F-\operatorname{coker} F=0$ (Nirenberg 1974, Vainberg and Trenogin 1974). For simplicity suppose that at certain g_L^2 $\dim\ker F=1$, and for a sufficiently small λ defined by $g^2\equiv g_L^2+\lambda$, $\dim\ker F=0$. So we can find small solutions in the form (Vainberg and Trenogin 1974)

$$\begin{pmatrix} B \\ \hat{A} \end{pmatrix}\equiv\begin{pmatrix} \sum_{i=1}^{\infty}\xi^iB_{i0}+\sum_{i=0}^{\infty}\xi^i\sum_{j=1}^{\infty}\lambda^jB_{ij} \\ \sum_{i=1}^{\infty}\xi^i\hat{A}_{i0}+\sum_{i=0}^{\infty}\xi^i\sum_{j=1}^{\infty}\lambda^j\hat{A}_{ij} \end{pmatrix} \tag{8}$$

where $\hat{A}_{ij}$, B_{ij} are unknown functions, $\lambda\equiv g^2-g_L^2$, ξ is a parameter. One can obtain $\hat{A}_{ij}$, B_{ij} after inserting equations (8) into equations (4) and equating terms with the same degree of λ, ξ. The linear operator F must be redefined in equations (4) beforehand, because it should be invertible. The validity of this expansion is proved by showing that the Lyapunov–Schmidt equations have non-trivial small solutions (for details see Vainberg and Trenogin (1974)).

The Lyapunov–Schmidt equation is

$$\sum_{i=2}^{\infty}L_{i0}\xi^i+\sum_{i=0}^{\infty}\xi^i\sum_{j=1}^{\infty}L_{ij}\lambda^j=0 \tag{9}$$

where

$$L_{ij}\equiv\int_\Omega \mathrm{d}\boldsymbol{x}\,(\tilde{B}_L,\tilde{A}_L)\begin{pmatrix} B_{ij} \\ \hat{A}_{ij} \end{pmatrix} \tag{10}$$

and $\begin{pmatrix} \tilde{B}_L \\ \tilde{A}_L \end{pmatrix}$ is a solution of the linearised equations

$$(-\nabla^2+g^2A_0^2)\tilde{B}+2g^2A_0\phi_0\tilde{A}=0 \qquad (\nabla^2-1/\rho^2+g^2\phi_0^2)\tilde{A}+2g^2A_0\phi_0\tilde{B}=0 \tag{11}$$

with $\begin{pmatrix} \tilde{B}_L \\ \tilde{A}_L \end{pmatrix}_{|\partial\Omega}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ($\partial\Omega$ is a boundary of a domain). Suppose that some of L_{ij} are non-zero. The solution $\xi(\lambda)$ to equation (9) is called small if $\lim_{\lambda\to 0}\xi(\lambda)=0$. Correspondingly, a small solution of equations (4) tends to zero as $\lambda\to 0$.

The Lyapunov–Schmidt theorem asserts that the small solutions to equation (9) and to the full nonlinear equations (equations (4) in our case) are in one-to-one correspondence, given by the expansion (8) (Vainberg and Trenogin 1974).

3. Main results

Suppose that our boundary conditions have the form

$$\begin{pmatrix} \phi \\ A \end{pmatrix}_{|\partial\Omega}=\begin{pmatrix} h(\partial\Omega) \\ 0 \end{pmatrix}. \tag{12a}$$

Then equations (2) admit a Coulomb solution (i.e. with the Abelian holonomy group)

$$A_0 = 0 \qquad B = \int_\Omega G*b + \int_{\partial\Omega} G_S*h \tag{12b}$$

where G, G_S are the relevant Green functions. We see, inserting (12b) into (11), that $\tilde{B} = 0$, but the second of equations (11) may have non-trivial solutions of class $C^{2+\mu}$ at least, everywhere except possibly along the 3-d axis. This is the Sturm–Liouville eigenvalue equation: there should exist a finite number of $g_k^2 < 0$ and an infinite number of eigenvalues

$$g_k^2 \to \infty \qquad (k \to \infty).$$

Assume that we deal with a simple eigenvalue g_L^2 with eigenfunction $f_L = \begin{pmatrix} 0 \\ f_L \end{pmatrix}$. Let $g^2 = g_L^2 + \lambda$ and define $\tilde{\Delta}$ by

$$\tilde{\Delta}\tilde{x} = -F\tilde{x} + \left(\int_\Omega \mathrm{d}\boldsymbol{x}\, f_L \tilde{x}_2\right) f_L \tag{13}$$

where $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$. The operator $\tilde{\Delta}$ is invertible, while F is not.

$$\tilde{\Delta} \text{ is a diagonal matrix,} \qquad \Delta \equiv \begin{pmatrix} \Delta, & 0 \\ 0, & \tilde{\Delta}_{22} \end{pmatrix}. \tag{14}$$

We now write equations (4) as follows:

$$\tilde{\Delta}\begin{pmatrix} B \\ \hat{A} \end{pmatrix} = (g_L^2 + \lambda)\begin{pmatrix} \hat{A}^2 B \\ B^2 \hat{A} \end{pmatrix} + \lambda\phi_0^2\begin{pmatrix} 0 \\ \hat{A} \end{pmatrix} + (g_L^2 + \lambda)\phi_0\begin{pmatrix} \hat{A}^2 \\ 2\hat{A}B \end{pmatrix} + \xi\begin{pmatrix} 0 \\ f_L \end{pmatrix} \tag{15}$$

where $\xi = \int_\Omega f_L \hat{A}\, \mathrm{d}\boldsymbol{x}$. The coefficient functions $\hat{A}_{ij}$, B_{ij} (see formula (8)) for the first few indices are the following:

$$\hat{A}_{20} = \hat{A}_{0i} = 0 \qquad (\forall i \in \mathbb{Z}) \qquad A_{10} = f_L \tag{16a}$$

$$\hat{A}_{11}(\boldsymbol{x}') = \int_\Omega \mathrm{d}\boldsymbol{x}\, \tilde{\Delta}_{22}^{-1}(\boldsymbol{x}', \boldsymbol{x})\phi_0^2(\boldsymbol{x}) f_L(\boldsymbol{x})$$

$$\hat{A}_{30}(\boldsymbol{x}') = \int_\Omega g_L^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \tilde{\Delta}_{22}^{-1}(\boldsymbol{x}', \boldsymbol{x}) f_L(\boldsymbol{x})\phi_0(\boldsymbol{x})\Delta^{-1}(\boldsymbol{x}, \boldsymbol{y})\phi_0(\boldsymbol{y}) f_L^2(\boldsymbol{y})$$

$$B_{0i} = 0 \qquad (\forall i \in \mathbb{Z}) \qquad B_{10} = B_{11} = B_{30} = 0 \tag{16b}$$

$$B_{30}(\boldsymbol{x}) = g_L^2 \int_\Omega \mathrm{d}\boldsymbol{y}\, \Delta^{-1}(\boldsymbol{x}, \boldsymbol{y})\phi_0(\boldsymbol{y}) f_L^2(\boldsymbol{y})$$

where Δ^{-1}, $\tilde{\Delta}_{22}^{-1}$ are the Green functions for the operators $\Delta = (\tilde{\Delta})_{11}$, $\tilde{\Delta}_{22} = (\tilde{\Delta})_{22}$. Hence

$$L_{10} = L_{20} = L_{0i} = 0 \qquad (\forall i \in \mathbb{Z})$$

$$L_{11} = \int_\Omega \mathrm{d}\boldsymbol{x}\, f_L(\boldsymbol{x})\hat{A}_{11}(\boldsymbol{x}) \qquad L_{30} = \int_\Omega \mathrm{d}\boldsymbol{x}\, f_L(\boldsymbol{x})\hat{A}_{30}(\boldsymbol{x}). \tag{17}$$

All L_{ij} are finite because of the invertibility of the operator $\tilde{\Delta}$. It may be proved (Vainberg and Trenogin 1974) that in this case the Lyapunov–Schmidt equations have three small solutions, which can be obtained approximately, by resolving

$$L_{11}\lambda\xi + L_{30}\xi^3 = 0. \tag{18}$$

Thence

$$\xi = 0 \qquad \text{or} \qquad \xi = \pm[(-L_{11}/L_{30})\lambda]^{1/2}. \tag{19}$$

It should be shown (Vainberg and Trenogin 1974) that the exact solution of equation (9) is now (besides $\xi = 0$)

$$\xi = \pm[(-L_{11}/L_{30})\lambda]^{1/2} + o(\sqrt{\lambda}) \qquad \text{where } o(\sqrt{\lambda})/\sqrt{\lambda} \to 0 \qquad (\lambda \to 0).$$

Hence we have a result (using the Lyapunov–Schmidt theorem): two new solutions branching from the old Coulomb potential at the point $g^2 = g_L^2$. The bifurcating solutions are approximated to first order by the zero-mode solution:

$$\begin{pmatrix}\phi\\ A\end{pmatrix} = \begin{pmatrix}\phi_0\\ 0\end{pmatrix} \pm \left(\frac{-L_{11}}{L_{30}}\lambda\right)^{1/2}\begin{pmatrix}0\\ f_L\end{pmatrix} + o(\sqrt{\lambda}). \tag{20}$$

Our bifurcation is overcritical; real solutions exist above g_L^2 because $-L_{11}/L_{30} > 0$ (see (16), (17) and use the fact that f_L is a fixed point of $\tilde{\Delta}$, and $\Delta^{-1} < 0$).

One can easily prove the non-Abelian nature of obtained solutions (see equation (1)) and occurrence of the magnetic field.

4. Example

The simplest possible case is the following: $b(\boldsymbol{x}) = 0$, boundary $\partial\Omega$ is a sphere of radius R, boundary conditions $\phi = \phi_0 = \text{constant}$, $A(\partial\Omega) = 0$, and $\phi(\infty) = A(\infty) = 0$. Then

$$\phi^3 = \begin{cases}\phi_0, & r \leqslant R\\ \phi_0 R/r, & r > R.\end{cases} \tag{21}$$

The bifurcating cylindrically symmetric solutions exist only inside the sphere of radius R:

$$\phi^3 = \begin{cases}\phi_0 + o(\sqrt{\lambda}), & r \leqslant R\\ \phi_0 R/r, & r > R\end{cases} \tag{22a}$$

$$A_i^1 = \begin{cases}\epsilon_{i3j}x_j \,\text{constant}\sqrt{\lambda}\,J_\nu(g_L\phi_0 r)r^{-1/2}, & r \leqslant R\\ 0, & r > R\end{cases} \tag{22b}$$

where $\nu = L + \frac{1}{2}$, J_ν are the Bessel functions, $g_L > 0$. All eigenvalues g_L^2 are simple. The potentials A_μ^a are of class C^∞ inside the sphere.

The eigenvalues are given (for $L = 1$) by

$$\tan(g_1\phi_0 R) = g_1\phi_0 R.$$

5. Conclusions

The main idea of this work is to show that near the Coulomb potential may exist highly non-trivial solutions, non-Abelian in their nature. That surprising phenomenon is important also because of its relation with stability. It may be shown that above the bifurcation point the Coulomb potential is unstable, while the bifurcating solutions are stable, at least under some assumptions about the nature of sources (e.g. for sources non-zero or analytic everywhere in the domain).

The methods presented above can be used to investigate the neighbourhood of other exact solutions to the Yang–Mills equations in Minkowski space (e.g. the Prasad–Sommerfeld–Bogomolny monopole or Julia–Zee dyon—see Actor (1979)). But because of the non-Abelian nature of the exact solutions it seems to be not so interesting as in the Coulomb case; the only new qualitative information which one should obtain is possibly an instability.

The bifurcation theory may give some information about the Gribov ambiguity (Actor (1979) and references therein). It is well known that for sufficiently weak potentials the Coulomb gauge exists locally (i.e. there is a certain neighbourhood of the unity group element where the gauge is unique). Some results obtained by many authors indicate that for strong potentials the gauge may not be unique even locally. This can be established with the help of the bifurcation theory. But note that the Gribov ambiguities include also phenomena of non-local nature (e.g. Gribov showed that even the potentials $A^a_\mu = 0$ have non-trivial gauge copy, far from the unity element).

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